Unit -III Differential calculus

Part –A

1. Define-Curvature and radius of curvature.

Ans:The rate of bending of the curve with respect to actual distance at'p' is called the curvature of the curve,which is denoted by 'k'.Therefore, $k=\delta_{S\to 0}\ \partial\psi/\partial s=d\psi/ds$.
Radius of curvature is reciprocal of curvature and it is denoted by ρ.therefore, $\rho=1/k$.

2. What is the curvature of $x^2+y^2-4x-6y+10=0$ at any point on it?

Ans: The given eqn is a circle, with radius $\sqrt{u^2+v^2-d}=\sqrt{2^2+3^2-10}=\sqrt{3}$.W.K.T for a circle $k=1/\text{radius}, k=1/\sqrt{3}$.

3. Find the radius of curvature at (3,-4) to the curve $x^2+y^2=25$.

Ans; The given eqn is a circle with radius r=5.therefore k=1/radius=1/5.
$\rho=1/k=1/5=5$.

4. Find the curvature of the curve $2x^2+2y^2+5x-2y+1=0$

Ans: $2x^2+2y^2+5x-2y+1=0$, $x^2+y^2+(5/2)x-y+(1/2)=0$, which is a eqn of a circle.
Therefore, $k=1/\text{radius}=1/(\sqrt{21}/4)=4/\sqrt{21}$

5. Find the radius of curvature at $x=\pi/2$ on the curve $y=4\sin x-\sin 2x$.

Ans: $y=4\sin x-\sin 2x$. $x=\pi/2$. Therefore, $4\sin(\pi/2)-\sin 2(\pi/2)=4$,Therefore, the point is $(\pi/2,4)$.
$\rho=[1+y']^{3/2}/y''$
$y'=dy/dx=4\cos x-2\cos 2x$,
$y'_{(\pi/2,4)}=2$,similarly,
$y''=d^2y/dx^2=-4\sin x+4\sin 2x$,
$y''_{(\pi/2,4)}=-4$
Therefore, $\rho=5\sqrt{5}/4$.

6. What is the curvature of a)straight line b)circle of radius 2 units

Ans: a)for straight lines k=0
b)for circle of radius 2 units,k=2

7.Find the radius of curvature of any point(x,y) on y=a log seec(x/a)

Ans: $y'=\tan(x/a)$,$y''=(1/a)\sec^2(x/a)$, $\rho=a.\sec(x/a)$.

8.Find the radius of curvature of the curve y=a cos h(x/a) at any point on it.

Ans: $y' = \sinh(x/a)$, $y'' = (1/a)\cosh(x/a)$,

$\rho = \{1+\sinh^2(x/a)\}^{3/2}/(1/a)\cosh(x/a)$

$= [\cosh^2(x/a)]^{3/2} / (1/a)\cosh(x/a)$

$= a.\cosh(x/a) = a.y^2/a^2 = y^2/a$

9.Find the radius of curvature at y=2a on the curve $y^2=4ax$

Ans: y=2a,x=a

Differentiating the given eqn,$2y.y'=4a$

$y'=4a/2y=2ay^{-1}$, $y'_{(a,2a)}=2a/2a=1$.

$y''= -2a/y^2$; $y''_{(a,2a)}=-2a/4a^2=-1/2a$

$\rho = [1+y']^{3/2} / y''$

$\rho_{(a,2a)}=\{1+1\}^{3/2}/(-1/2a) = 2a.2^{3/2} = 2^{5/2}.a$

10.For the curve $x^2 = 2c(y-c)$,find the radius of curvature at (0,c).

Ans: $x^2 = 2c(y-c)$.-------(1).

Differentiate with respect to x.

$2x = 2cy'$.

$y' = (x/c)$.

$y'' = 1/c$.

Find the envelope of the family of straight lines $x\cos\alpha + y\sin\alpha = P$,where α is the parameter.

$\rho = [1+y']^{3/2} / y''$

$= [1 + (x/c)]^{3/2} / (1/c)$.

$P_{(0,c)} = c$.

11.Write the formula for radius of curvature in Cartesian form, parametric form, and polar form.

Ans:

(i) Cartesian form :

$$\rho = [1+y']^{3/2} / y''$$

(ii) Parametric form :

$$\rho = [x'^2+y'^2]^{3/2} / [x'y''- x''y']$$

(iii) polar form :

$$\rho = [r'^2+r^2]^{3/2} / [r^2 + 2r'-rr''].$$

12.Find the envelope of the family of straight lines $y = mx \pm (m^2 -1)^{1/2}$,where m is the parameter

Ans: $y = mx \pm (m^2 -1)^{1/2}$.

$m^2 -1 = y^2 + m^2x^2-2mxy$.

$(x^2-1) m^2 – 2xym + y^2 +1 = 0$.

The envelope is given by equation

$$4 x^2 y^2 \text{-} 4(x^{2-1}) (y^2+1) = 0.$$

$$(x^2 /1)- (y^2 / 1) = 1.$$

13. Find the envelope of the family of straight lines $y = mx + (a/m)$.
,where m is the parameter

Ans: $y = mx + (a/m)$.

$y = mx + (a/m)$.

$m^2x - my + a = 0$.

The envelope is given by equation.,$y^2 - 4ax = 0$, which is a parabola.

14. Find the envelope of the family of straight lines $y = mx + am^2$, where m is the parameter

Ans: $y = mx + am^2$.

$am^2 + mx - y = 0$.

The envelope is given by $x^2 + 4xy = 0$.

15. Find the envelope of the family of straight lines $x\cos\alpha + y\sin\alpha = P$,where α is the parameter.

Ans: $x\cos\alpha + y\sin\alpha = P$ -----------(1)

Diffrentiate (1) patialy with respect to α ,

$-x\sin\alpha + y\cos\alpha = 0$.-----------(2)

$(1)^2 + (2)^2$ gives,$x^2+y^2=p^2$.

16. Find the envelope of the family of circles $(x-a)^2+y^2 = 4a$.

Ans: $(x-a)^2+y^2 = 4a$.---------(1)

Diffrentiate (1) patialy with respect to α ,

$2(x-a)(-1)+0 = 4$.

$a = x+2$.

(1) Implies $y^2-4x = 4$.

17. Find the envelope of the family of straight lines $x\cos\alpha + y\sin\alpha = a\sec\alpha$,where α is the parameter.

Ans: $x\cos\alpha + y\sin\alpha = a\sec\alpha$.

Divide by $\cos\alpha$

$x + y\tan\alpha = a\sec^2\alpha$.

$a\tan^2\alpha - y\tan\alpha + (a-x) = 0$.

The envelope is given by $y^2 = 4a(a-x)$.

18. Find the envelope of $(x/t) + yt = 2c$, t is the parameter.

Ans: $(x/t) + yt = 2c$

$yt^2+x = 2ct$.

$yt^2+x - 2ct =.0$.

The envelope is given by $xy = c^2$.

19. Show that the family of circles $(x-a)^2+y^2=a^2$, a is the parameter has no envelope.

Ans: $(x-a)^2+y^2=a^2$.-------(1)
20.Diffrentiate (1) patialy with respect to α ,
$-2(x-a) = 2a$.
$x = 2a$.
Therefore $y = 0$.

20.If the centre of curvature is $((c/a) \cos^3 t, (c/a) \sin^3 t)$,find the evolute of curve.

Ans: $\overline{x} = (c/a)\cos^3 t$, $\overline{y} = (c/a)\sin^3 t$,

$$(ax)^{2/3}+(ay)^{2/3} = c^{2/3}.$$

Part -B

1.Find the radius of curvature at t on $x = e^t$ on $x= e^t \cos t$, $x= e^t \sin t$.

2.Find the evolute of the rectangular hyperbola $xy = c^2$.

3.Find the equation of the envelope of $(x/a)+(y/b)=1$,where the parameters a and are connected by the relation $a^2+b^2=c^2$ and c is the a constant.

4. Find the equation of the circle of curvature at $(a/4,a/4)$ for the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$

5. Find the evolute of of the curve $x=a(\cos\theta+\theta\sin\theta)$,$y= a(\sin\theta-\theta\cos\theta)$.

6. Find the radius of curvature at (1,1) to the curve $x^{2/3}+y^{2/3}=2$.

7.Prove that the envelope of the family of straight lines $y= mx-2am-am^3$ is $27ay^2 = 4(x-2a)^3$.

8.find the locus of the centre of curvature of parabola $y^2 = 4ax$.

9. Prove that envelope of $(x/a)+(y/b)=1$,where the parameters a and are connected by the relation $a+b=c$ is $\sqrt{x}+\sqrt{y}=\sqrt{c}$.

10.In the curve $\sqrt{(x/a)}+ \sqrt{(y/b)} =1$, show that the radius of curvature at the point (x,y) varies as $(ax+by)^{3/2}$.

11.Show that the evolute of the hyperbola $(x^2/a^2)+(y^2/b^2) =1$ is $(ax)^{2/3} -(by)^{2/3}= (a^2+b^2)^{2/3}$.

12. Find the radius of curvature at the point (r, θ) on the curve $r = a \cos \theta$.

13. Find the evolute of the parabola $y^2= 4ax$ considering it as the envelope of its normals.

14. Find the equation of the envelope of $(x^2/a^2)+(y^2/b^2)=1$,where the parameters a and are connected by the relation $a+b=c$.

15.If ρ is the radius of curvature at any point (x,y) on the curve $y = (ax / (a+x))$ prove that $(2\rho/a)^{2/3}=(x/y)^2+(y/x)^2$.

16. Find the radius of curvature of the curve $r = a (1+\cos \theta)$ at $\theta = \pi/2$.

17.If centre of the curvature of ellipse $(x^2/a^2)+(y^2/b^2)=1$ at one end of the minor axes lies at the other end.Prove that the
eccentricity of the ellipse is $1/\sqrt{2}$

18.Prove that the evolute of the curve $x= a(\cos\theta+\log(\tan \theta/2))$,$y= a \sin \theta$ is the catenary $y=a\cosh(x/a)$.

19.Find the envelope of $(ax/\cos\theta)-(by/\sin\theta) = (a^2-b^2)$ where θ is the parameter.

20.Find the radius of curvature at the point $\theta=0$ on the curve $r= ae^{\theta\cot\alpha}$.